Price Discovery in a Matching and Bargaining Market with Aggregate Uncertainty

Artyom Shneyerov[1] and Adam Chi Leung Wong[2]

Workshop in Memory of Artyom Shneyerov
CIRANO
October 12, 2018

[1]Concordia University and CIREQ, CIRANO
[2]Lingnan University

- In a market where buyers and sellers are strategic and **uncertain about demand and supply**, at what price should they trade?
- Study dynamic market with **search frictions** and **decentralized bilateral bargaining**
  - e.g. second-hand housing market, used car market, labor market
- 2 states:
  - H: high-demand low-supply (sellers' market)
  - L: high-supply low-demand (buyers' market)
- Traders learn from search experiences
- If search frictions are small, would the transaction prices be close to the true-state Walrasian (or competitive, or market-clearing) price?

Main Results

In our model, as search frictions converge to 0, the market discovers the true-state Walrasian price quickly:

- transaction prices converge to the true-state Walrasian price in expectation
- the rate of convergence is linear in search frictions, the same as it would be if the state were commonly known

Literature (Dynamic matching and bargaining games)

- Initiated by Rubinstein & Wolinsky (1985), homogeneous buyers/sellers, no uncertainty
- Heterogeneous buyers/sellers, complete info bargaining
  - Gale (1987), Mortensen & Wright (2002)
- Heterogeneous buyers/sellers, IPV bargaining
  - Wolinsky (1988), Satterthwaite & Shneyerov (2007, 2008), Atakan (2008, 2009), Shneyerov & Wong (2010a,b), Lauermann (2012, 2013)
- Common values uncertainty
  - Wolinsky (1990), Blouin & Serrano (2001), Serrano (2002)
- Aggregate (demand-supply) uncertainty
  - Majumdar, Shneyerov, & Xie (2016), Lauermann, Merzyn, & Virag (2018)

Model

- Buyers/sellers arrive at market deterministically and continuously
- Each seller has a unit supply of a homogeneous, indivisible good; cost is 0
- Each buyer has a unit demand; valuation is 1

Model

- Buyers/sellers arrive at market deterministically and continuously
- Each seller has a unit supply of a homogeneous, indivisible good; cost is 0
- Each buyer has a unit demand; valuation is 1
- Two possible states: $\omega \in \{H, L\}$; inflow rates of buyers/sellers in state ω are λ_B^ω and λ_S^ω

Assumption 1: $\lambda_B^H > \lambda_S^H$ and $\lambda_B^L < \lambda_S^L$.

- State is constant over time. No one knows the true state; common prior belief ϕ^ω

Note: flow Walrasian price is 1 if $\omega = H$ and 0 if $\omega = L$

Model

- Buyers/sellers arrive at market deterministically and continuously
- Each seller has a unit supply of a homogeneous, indivisible good; cost is 0
- Each buyer has a unit demand; valuation is 1
- Two possible states: $\omega \in \{H, L\}$; inflow rates of buyers/sellers in state ω are λ_B^ω and λ_S^ω

Assumption 1: $\lambda_B^H > \lambda_S^H$ and $\lambda_B^L < \lambda_S^L$.

- State is constant over time. No one knows the true state; common prior belief ϕ^ω

Note: flow Walrasian price is 1 if $\omega = H$ and 0 if $\omega = L$

- Every trader is risk neutral
- Continuous time, infinite horizon; focus on steady state

- Given stocks of buyers/sellers Λ_B, Λ_S, the mass of pairs matched per unit time is $\mu \cdot \min\{\Lambda_B, \Lambda_S\}$
- Who gets matched and Who matches whom are random
- Once matched, they bargain:
  1. Nature randomly chooses a proposer: buyer with prob. $\beta_B \in (0,1)$; seller with prob. $\beta_S \equiv 1 - \beta_B$
  2. Proposer makes take-it-or-leave-it price offer
  3. Responder chooses to accept or reject

- Given stocks of buyers/sellers Λ_B, Λ_S, the mass of pairs matched per unit time is $\mu \cdot \min\{\Lambda_B, \Lambda_S\}$
- Who gets matched and Who matches whom are random
- Once matched, they bargain:
  1. Nature randomly chooses a proposer: buyer with prob. $\beta_B \in (0, 1)$; seller with prob. $\beta_S \equiv 1 - \beta_B$
  2. Proposer makes take-it-or-leave-it price offer
  3. Responder chooses to accept or reject

Assumption 2: Upon meeting, each trader observes the total time his partner has participated in the market.

- If trade at p, buyer leaves with payoff $1 - p$, seller leaves with p
- If don't trade, stay searching for another match
- Friction profile: (r, δ)
  - $\delta > 0$: exogenous exit rate
  - $r \geq 0$: time discount rate

Full trade (steady state) market equilibrium

Basic equilibrium objects:

- steady state stocks and distributions of traders
- traders' beliefs about state
- traders' bargaining strategies

Full trade (steady state) market equilibrium

Basic equilibrium objects:

- steady state stocks and distributions of traders
- traders' beliefs about state
- traders' bargaining strategies

such that

- Given bargaining strategies, steady state equations are satisfied to maintain the stocks and distributions
- Given steady state stocks and distributions, the traders' beliefs and bargaining strategies constitute Perfect Bayesian Equilibrium
- In addition, restrict attention to *full trade equilibria* (FTE), in which every meeting on equilibrium path results in trade.

Steady state stocks

For each $\omega = L, H$, stocks $\Lambda_B^\omega, \Lambda_S^\omega$ satisfy

$$\lambda_B^\omega = \delta \Lambda_B^\omega + \mu \min\{\Lambda_B^\omega, \Lambda_S^\omega\}$$

$$\lambda_S^\omega = \delta \Lambda_S^\omega + \mu \min\{\Lambda_B^\omega, \Lambda_S^\omega\}$$

so that

$$\Lambda_B^\omega = \frac{(\delta + \mu)\lambda_B^\omega - \mu \min\{\lambda_B^\omega, \lambda_S^\omega\}}{\delta(\delta + \mu)},$$

$$\Lambda_S^\omega = \frac{(\delta + \mu)\lambda_S^\omega - \mu \min\{\lambda_B^\omega, \lambda_S^\omega\}}{\delta(\delta + \mu)}.$$

Note: $\Lambda_B^H > \Lambda_S^H$ and $\Lambda_B^L < \Lambda_S^L$.

Steady state finding rates

For each $\omega = L, H$, finding rates $\alpha_B^\omega, \alpha_S^\omega$ are

$$\alpha_B^\omega \equiv \frac{\mu \min\{\Lambda_B^\omega, \Lambda_S^\omega\}}{\Lambda_B^\omega}, \quad \alpha_S^\omega \equiv \frac{\mu \min\{\Lambda_B^\omega, \Lambda_S^\omega\}}{\Lambda_S^\omega}$$

Steady state finding rates

For each $\omega = L, H$, finding rates $\alpha_B^\omega, \alpha_S^\omega$ are

$$\alpha_B^\omega \equiv \frac{\mu \min\{\Lambda_B^\omega, \Lambda_S^\omega\}}{\Lambda_B^\omega}, \quad \alpha_S^\omega \equiv \frac{\mu \min\{\Lambda_B^\omega, \Lambda_S^\omega\}}{\Lambda_S^\omega}$$

In particular, short sides' finding rates are

$$\alpha_B^L = \alpha_S^H = \mu,$$

long sides' finding rates are

$$\alpha_B^H = \frac{\delta\mu\lambda_S^H}{(\delta+\mu)\lambda_B^H - \mu\lambda_S^H} < \mu,$$

$$\alpha_S^L = \frac{\delta\mu\lambda_B^L}{(\delta+\mu)\lambda_S^L - \mu\lambda_B^L} < \mu.$$

Lemma 1. α_B^H and α_S^L are $O(\delta)$.

Steady state distributions

- Let $G_B^\omega(t_B)$ be the fraction of buyers' steady-state stock in state ω who have been in the market for less than time t_B
- Steady-state equation for $G_B^\omega(\cdot)$ implies

$$G_B^\omega(t_B) = 1 - \exp(-(\delta + \alpha_B^\omega)t_B)$$

Steady state distributions

- Let $G_B^\omega(t_B)$ be the fraction of buyers' steady-state stock in state ω who have been in the market for less than time t_B
- Steady-state equation for $G_B^\omega(\cdot)$ implies

$$G_B^\omega(t_B) = 1 - \exp(-(\delta + \alpha_B^\omega)t_B)$$

Alternative Interpretation: conditional distribution of searching time

- $G_B^\omega(t_B)$ is, from an unmatched buyer's perspective, the prob. of being matched after some searching time less than t_B, conditional on the event that the true state is ω and this buyer will meet a seller (rather than exogenously exit before meeting)
- Similar note for $G_S^\omega(t_S) = 1 - \exp(-(\delta + \alpha_S^\omega)t_S)$

Belief formation

Search history and bargaining history

Search history (on or off equilibrium path) of a buyer who has met n sellers:

$$(t_{B1}, \ldots, t_{Bn}, t_{B(n+1)}; t_{S1}, \ldots, t_{Sn})$$

- t_{Bi} for $i \in \{1, \ldots, n\}$ is searching time spent to have the i-th meeting
- t_{Si} for $i \in \{1, \ldots, n\}$ is the observed time on the market of the i-th seller met
- $t_{B(n+1)}$ is the time on the market since last meeting

Belief formation

Search history and bargaining history

Search history (on or off equilibrium path) of a buyer who has met n sellers:

$$(t_{B1}, \ldots, t_{Bn}, t_{B(n+1)}; t_{S1}, \ldots, t_{Sn})$$

- t_{Bi} for $i \in \{1, \ldots, n\}$ is searching time spent to have the i-th meeting
- t_{Si} for $i \in \{1, \ldots, n\}$ is the observed time on the market of the i-th seller met
- $t_{B(n+1)}$ is the time on the market since last meeting

Bargaining history:

- which side proposed in previous meetings
- previous price offers
- that these offers are rejected

Belief formation

Search history and bargaining history

Search history (on or off equilibrium path) of a buyer who has met n sellers:

$$(t_{B1}, \dots, t_{Bn}, t_{B(n+1)}; t_{S1}, \dots, t_{Sn})$$

- t_{Bi} for $i \in \{1, \dots, n\}$ is searching time spent to have the i-th meeting
- t_{Si} for $i \in \{1, \dots, n\}$ is the observed time on the market of the i-th seller met
- $t_{B(n+1)}$ is the time on the market since last meeting

Bargaining history:

- which side proposed in previous meetings
- previous price offers
- that these offers are rejected

Can WLOG assume every trader only uses search history to update belief, since focus on FTE.

Belief formation

Updating from search history

$$h_B \equiv (t_{B1}, \ldots, t_{Bn}, t_{B(n+1)}; t_{S1}, \ldots, t_{Sn})$$

- Given α_B^ω, α_S^ω, $G_B^\omega(t_B)$, $G_S^\omega(t_S)$, a buyer's belief $\pi_B^\omega(h_B)$ about state ω after h_B can be computed from Bayes' rule
- $\pi_B^\omega(h_B)$ depends on h_B only through $\sum_{i=1}^{n+1} t_{Bi} \equiv t_B$, $\sum_{i=1}^{n} t_{Si} \equiv t_S$ and n

Belief formation

Updating from search history

$$h_B \equiv (t_{B1}, \ldots, t_{Bn}, t_{B(n+1)}; t_{S1}, \ldots, t_{Sn})$$

- Given α_B^ω, α_S^ω, $G_B^\omega(t_B)$, $G_S^\omega(t_S)$, a buyer's belief $\pi_B^\omega(h_B)$ about state ω after h_B can be computed from Bayes' rule
- $\pi_B^\omega(h_B)$ depends on h_B only through $\sum_{i=1}^{n+1} t_{Bi} \equiv t_B$, $\sum_{i=1}^{n} t_{Si} \equiv t_S$ and n
- Similarly, $\pi_S^\omega(h_S)$ depends on h_S only through $\sum_{i=1}^{n} t_{Bi} \equiv t_B$, $\sum_{i=1}^{n+1} t_{Si} \equiv t_S$ and n
- Write $\pi_B^\omega(t_B, t_S, n)$ and $\pi_S^\omega(t_B, t_S, n)$

Belief formation

Updating from search history

$$h_B \equiv (t_{B1}, \ldots, t_{Bn}, t_{B(n+1)}; t_{S1}, \ldots, t_{Sn})$$

- Given α_B^ω, α_S^ω, $G_B^\omega(t_B)$, $G_S^\omega(t_S)$, a buyer's belief $\pi_B^\omega(h_B)$ about state ω after h_B can be computed from Bayes' rule
- $\pi_B^\omega(h_B)$ depends on h_B only through $\sum_{i=1}^{n+1} t_{Bi} \equiv t_B$, $\sum_{i=1}^{n} t_{Si} \equiv t_S$ and n
- Similarly, $\pi_S^\omega(h_S)$ depends on h_S only through $\sum_{i=1}^{n} t_{Bi} \equiv t_B$, $\sum_{i=1}^{n+1} t_{Si} \equiv t_S$ and n
- Write $\pi_B^\omega(t_B, t_S, n)$ and $\pi_S^\omega(t_B, t_S, n)$

Feature: $\pi_B^\omega(t_B, t_S, 1) = \pi_S^\omega(t_B, t_S, 1)$ for every t_B, t_S

- meeting on eqm path is the first meeting for both
- bargaining on eqm path is under sym info

Bellman equations

- Bargaining strategies are fully characterized by the continuation payoffs (or search values) $W_B(h_B)$ and $W_S(h_S)$ just after breaking-up
- Write $W_B(t_B, t_S, n)$ and $W_S(t_B, t_S, n)$

Bellman equations

- Bargaining strategies are fully characterized by the continuation payoffs (or search values) $W_B(h_B)$ and $W_S(h_S)$ just after breaking-up
- Write $W_B(t_B, t_S, n)$ and $W_S(t_B, t_S, n)$

Let T_B, T_S be independent r.v. that follow distributions $G_B^\omega(\cdot)$, $G_S^\omega(\cdot)$.

$$W_B(t_B, t_S, n) = \sum_{\omega=L,H} \pi_B^\omega(t_B, t_S, n) \frac{\alpha_B^\omega}{\delta + \alpha_B^\omega} \mathbb{E}[e^{-rT_B} q_B(t_B + T_B, t_S, n; T_S)|\omega]$$

where $q_B(t_B + T_B, t_S, n; T_S) \equiv$

$$\beta_B \max\{1 - W_S(t_B + T_B, T_S, 1), W_B(t_B + T_B, t_S + T_S, n + 1)\}$$
$$+ \beta_S \max\{W_B(t_B + T_B, T_S, 1), W_B(t_B + T_B, t_S + T_S, n + 1)\}$$

Similarly for $W_S(t_B, t_S, n)$

Equilibrium

- Given $\alpha_B^\omega, \alpha_S^\omega, G_B^\omega(\cdot), G_S^\omega(\cdot), \pi_B^\omega(\cdot), \pi_S^\omega(\cdot)$ derived above, *full trade (market) equilibrium (FTE)* can be redefined as functions

$$W_B, W_S : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N} \to [0,1]$$

  that solve buyers' and sellers' Bellman equations and such that the *trading condition*

$$W_B(t_B, t_S, 1) + W_S(t_B, t_S, 1) \leq 1$$

  holds for every (t_B, t_S).
- Transaction prices on equilibrium path are:
  - either $W_S(t_B, t_S, 1)$ when buyer proposes
  - or $1 - W_B(t_B, t_S, 1)$ when seller proposes

No uncertainty benchmark

Existence, uniqueness, rate of convergence under certainty

Suppose true state ω is commonly known ($\phi^\omega = 1$).

- W_B, W_S become constants

$$\overline{W}_B^\omega = \frac{\beta_B \alpha_B^\omega}{r + \delta + \beta_B \alpha_B^\omega + \beta_S \alpha_S^\omega},$$
$$\overline{W}_S^\omega = \frac{\beta_S \alpha_S^\omega}{r + \delta + \beta_B \alpha_B^\omega + \beta_S \alpha_S^\omega}.$$

- $\overline{W}_B^\omega + \overline{W}_S^\omega < 1$
- $\overline{W}_B^H, 1 - \overline{W}_S^H, 1 - \overline{W}_B^L, \overline{W}_S^L = O(r + \delta)$
  - because $\alpha_B^L = \alpha_S^H = \mu$ and $\alpha_B^H, \alpha_S^L = O(\delta)$

No uncertainty benchmark

Existence, uniqueness, rate of convergence under certainty

Proposition 1. If true state ω is commonly known,

- $\forall (r, \delta) \in \mathbb{R}_+ \times \mathbb{R}_{++}$, $\exists$ a unique FTE.
- $\exists \overline{C}_0, \overline{C}_1 > 0$, not depending on r, δ, s.t. when $r + \delta > 0$ is sufficiently small,

$$\overline{C}_0 \cdot (r+\delta) \leq \begin{matrix} \overline{W}_B^H, \\ 1 - \overline{W}_S^H, \\ 1 - \overline{W}_B^L, \\ \overline{W}_S^L \end{matrix} \leq \overline{C}_1 \cdot (r+\delta),$$

i.e., **discrepancy between equilibrium transaction prices and Walrasian price is of order $r + \delta$.**

Uniqueness

Return to the aggregate uncertainty case $(\phi^L, \phi^H \in (0,1))$

- Neglect the trading condition: FTE *candidate* defined only by a pair of Bellman equations

Proposition 2 (Uniqueness). $\forall (r, \delta) \in \mathbb{R}_+ \times \mathbb{R}_{++}$, there is at most one FTE.

Sketch of proof: Apply Contraction Mapping Theorem to show that the system of Bellman equations has a unique solution.

Basic equilibrium properties

Proposition 3. In any FTE,

- $\pi_B^L(t_B, t_S, n)$ and $W_B(t_B, t_S, n)$ are continuous in (t_B, t_S), nonincreasing in t_B, and nondecreasing in t_S;
- $\pi_S^H(t_B, t_S, n)$ and $W_S(t_B, t_S, n)$ are continuous in (t_B, t_S), nondecreasing in t_B, and nonincreasing in t_S;
- $\forall (t_B, t_S, n) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N}$,

$$\overline{W}_B^H \leq W_B(t_B, t_S, n) \leq \overline{W}_B^L,$$

$$\overline{W}_S^L \leq W_S(t_B, t_S, n) \leq \overline{W}_S^H.$$

Belief convergence

- Traders' bargaining values (on equilibrium path) depend on their outside option values.
- Their outside option values depend on their first-order beliefs and their bargaining values of off-equilibrium future bargaining.
- Values of off-equilibrium future bargaining depend on second-level outside option values, which in turn depend on second-order beliefs and bargaining values of second-level off-equilibrium future bargaining; and so on.
- In a off-equilibrium bargaining, buyer and seller do not have symmetric info; one or both of their beliefs are formed based on wrong info about n
- However, all these on- and off-equilibrium beliefs become asymptotically precise in expectation.

Belief convergence

Let T_{Bi}'s and T_{Si}'s be *independent* random copies of T_B and T_S respectively.

Lemma 3. For $j = B, S$,

$$\max_{1\leq k_1,k_2,k_3\leq n}\left\{\mathbb{E}\left[\pi_j^L\left(\sum_{i=1}^{k_1} T_{Bi}, \sum_{i=1}^{k_2} T_{Si}, k_3\right)|H\right]\right\} \leq (c_1 + c_2 n)\cdot\delta,$$

$$\max_{1\leq k_1,k_2,k_3\leq n}\left\{\mathbb{E}\left[\pi_j^H\left(\sum_{i=1}^{k_1} T_{Bi}, \sum_{i=1}^{k_2} T_{Si}, k_3\right)|L\right]\right\} \leq (c_1 + c_2 n)\cdot\delta,$$

where c_1, c_2 are constants not depending on r, δ, n.

Intuition:

- Say true state is H, and let $\delta \to 0$.
- Recall that $\alpha_S^H = \mu$ but $\alpha_B^H = O(\delta)$.
- Buyers' random searching time $T_B \to \infty$ in probability, but T_S does not.
- The reverse is true if true state is L.
- Realizations of T_B, T_S are more and more informative as $\delta \to 0$.

Convergence of prices

To no uncertainty benchmark

Proposition 4. In any FTE,

$$0 \leq \begin{array}{c} \mathbb{E}\left[W_B(T_B, T_S, 1)|H\right] - \overline{W}_B^H, \\ \overline{W}_S^H - \mathbb{E}\left[W_S(T_B, T_S, 1)|H\right], \\ \overline{W}_B^L - \mathbb{E}\left[W_B(T_B, T_S, 1)|L\right], \\ \mathbb{E}\left[W_S(T_B, T_S, 1)|L\right] - \overline{W}_S^L \end{array} \leq C \cdot \delta,$$

where C is a constant that does not depend on r, δ.

- Convergence in expectation (Recall that $\forall (t_B, t_S)$ $\overline{W}_B^H \leq W_B(t_B, t_S, 1) \leq \overline{W}_B^L$ and $\overline{W}_S^L \leq W_S(t_B, t_S, 1) \leq \overline{W}_S^H$)
- **expected discrepancy between equilibrium transaction prices and true-state no uncertainty benchmark price is of order δ.**

Convergence of prices

To true-state Walrasian price

Main Theorem: $\exists$ constants $C_0, C_1 > 0$ not depending on r, δ s.t. if $r + \delta > 0$ is sufficiently small, any FTE satisfies

$$C_0 \cdot (r+\delta) \leq \begin{array}{c} \mathbb{E}\left[W_B\left(T_B, T_S, 1\right)|H\right], \\ 1-\mathbb{E}\left[W_S\left(T_B, T_S, 1\right)|H\right], \\ 1-\mathbb{E}\left[W_B\left(T_B, T_S, 1\right)|L\right], \\ \mathbb{E}\left[W_S\left(T_B, T_S, 1\right)|L\right] \end{array} \leq C_1 \cdot (r+\delta),$$

i.e., **expected discrepancy between equilibrium transaction prices and the true-state Walrasian price is of order $r + \delta$.**

Existence

Proposition 5. $\forall \underline{r} > 0$, $\exists \bar{\delta} > 0$ s.t.
whenever $r \geq \underline{r}$ and $0 < \delta \leq \bar{\delta}$, the FTE candidate satisfies

$$W_B(t_B, t_S, 1) + W_S(t_B, t_S, 1) \leq 1 \quad \forall (t_B, t_S) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

Corollary 3. For any level $\tau > 0$, $\exists (r, \delta) \in \mathbb{R}_+ \times \mathbb{R}_{++}$ with $r + \delta = \tau$ s.t. a FTE exists under (r, δ).

Summary

- Study dynamic model of a market with search friction and bilateral random-proposer take-it-or-leave-it bargaining
- Two possible states:
  - at H state, more buyers than sellers
  - at L state, more sellers than buyers
- The only info transmitted in a meeting is the time a trader spent on the market
- As search frictions vanish, the market discovers the true-state competitive price quickly
  - Transaction prices converge to the true-state Walrasian price in expectation
  - Rate of convergence is linear in the total search friction, the same as it would be if the state were commonly known.